Operations with Power Series

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

These operations can change the interval of convergence for the resulting series.

Recall that $\dfrac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$ represents the sum of a convergent geometric series where $|r| < 1$. $-1 < r < 1$

$f(x) = \dfrac{6}{7-(x-0)}$, $c = 0$

1. Find a geometric power series for the function, centered at 0.

a. $f(x) = \dfrac{6/7}{\frac{7}{7} - \frac{x}{7}}$

$f(x) = \dfrac{6/7}{1 - \frac{x}{7}}$

IOC: $\left|\frac{x}{7}\right| < 1 \to -1 < \frac{x}{7} < 1 \to -7 < x < 7$

So, $f(x) = \sum_{n=0}^{\infty} ar^n = \boxed{\sum_{n=0}^{\infty} \frac{6}{7}\left(\frac{x}{7}\right)^n, \ (-7, 7)}$

b. $f(x) = \dfrac{1}{1+x}$

$$\begin{array}{r} 1 - x + x^2 - x^3 \\ (1+x)\overline{)\,1 + 0 + 0 + 0 + 0} \\ -(1+x) \\ \hline -x + 0 \\ -(-x - x^2) \\ \hline x^2 + 0 \\ -(x^2 + x^3) \\ \hline -x^3 + 0 \\ -(-x^3 - x^4) \\ \hline x^4 \end{array}$$

$f(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + - \ldots$

$\boxed{f(x) = \sum_{n=0}^{\infty} (-x)^n, \ (-1, 1)}$

IOC: $|-x| < 1 \to |x| < 1 \to -1 < x < 1$

2. Find a geometric power series for the function, centered at c, and determine the interval of convergence.

$x - c \to$ centered at c
$x-(-2) \to$ centered at -2

a. $f(x) = \dfrac{4}{5-x}, \quad c = -2$

$f(x) = \sum_{n=0}^{\infty} ar^n$

$f(x) = \dfrac{4}{(5+2)-(x-(-2))}$

$f(x) = \dfrac{4/7}{\frac{7}{7}-\frac{(x+2)}{7}}$

$f(x) = \dfrac{4/7}{1-\frac{x+2}{7}}$

$$\boxed{f(x) = \sum_{n=0}^{\infty}\left(\frac{4}{7}\right)\left(\frac{x+2}{7}\right)^n, \ (-9,5)}$$

I.O.C.:

$\left|\frac{x+2}{7}\right| < 1 \to -1 < \frac{x+2}{7} < 1 \to -7 < x+2 < 7 \to -9 < x < 5$

b. $f(x) = \dfrac{1}{2x-5}, \quad c = 2$

$f(x) = \sum_{n=0}^{\infty} ar^n$

$f(x) = \dfrac{1}{-5-(-2x)}$

$f(x) = \dfrac{1}{(-5+4)-[-2(x-2)]}$

$f(x) = \dfrac{1/-1}{\frac{-1}{-1}-\frac{[-2(x-2)]}{-1}}$

$f(x) = \dfrac{-1}{1-2(x-2)}$

$$\boxed{f(x) = \sum_{n=0}^{\infty}(-1)[2(x-2)]^n, \ \left(\frac{3}{2},\frac{5}{2}\right)}$$

IOC:

$|2(x-2)| < 1 \to -1 < 2(x-2) < 1 \to -\frac{1}{2} < x-2 < \frac{1}{2} \to \frac{3}{2} < x < \frac{5}{2}$

c. $f(x)=\dfrac{4}{4+x^2}$, $c=0$

$f(x^N)=\sum_{n=0}^{\infty} a_n (x^n)^N$

Consider $g(x)=\dfrac{4}{4+x}$, $c=0$

so $f(x)=g(x^2)$

$g(x)=\dfrac{4}{4+x}$, $c=0$

$g(x)=\dfrac{4/4}{\frac{4}{4}-\left(\frac{-x}{4}\right)}$

$g(x)=\sum_{n=0}^{\infty}(1)\left(-\frac{x}{4}\right)^n$

$g(x)=\dfrac{1}{1-\left(-\frac{x}{4}\right)}$, $(-4,4)$

IOC:

$\left|-\frac{x}{4}\right|<1 \to \left|\frac{x}{4}\right|<1 \to -4<x<4$

$g(x^2)=f(x)$

$f(x)=\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n (x^n)^2$

$\boxed{f(x)=\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n x^{2n},\ (-2,2)}$

for x, our IOC was $(-4,4)$

for x^2, $(-2)^2=4$ and $(2)^2=4$

$\left(-\frac{1}{4}\right)^n \cdot 4^n=\left(-\frac{1}{4}\cdot 4\right)^n=-1$

3. Use the power series $\dfrac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$ to determine a power series, centered at 0, for the function. Identify the interval of convergence.

a. $h(x)=\dfrac{x}{x^2-1}=\dfrac{1}{2(1-x)}+\dfrac{1}{2(1+x)}$

$h(x)=\frac{1}{2}\left[\frac{1}{1-x}+\frac{1}{1+x}\right]$

$h(x)=\frac{1}{2}\left[\sum_{n=0}^{\infty}(1)x^n+\sum_{n=0}^{\infty}(-1)^n x^n\right]$

$h(x)=\frac{1}{2}\sum_{n=0}^{\infty}(1+(-1)^n)x^n=\frac{1}{2}\left((1+(-1)^0)x^0+(1+(-1)^1)x^1+(1+(-1)^2)x^2+(1+(-1)^3)x^3+\cdots\right)$

$=\frac{1}{2}\left[2x^0+0x^1+2x^2+0x^3+\cdots\right]$

$=1+x^2+x^4+x^6+\cdots$

$$\boxed{h(x) = \sum_{n=0}^{\infty} x^{2n}, \quad (-1, 1)}$$

<u>Theorem</u>: Properties of functions defined by power series (from 9.8)

If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has a radius of convergence of $R>0$, then, on the interval $(c-R, c+R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative are as follows:

1) $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

2) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1 \frac{(x-c)^2}{2} + a_2 \frac{(x-c)^3}{3} + \cdots$

b. $f(x)=\ln\left(1-x^2\right)=\int\frac{1}{1+x}dx-\int\frac{1}{1-x}dx$

$$f(x)=\int\sum_{n=0}^{\infty}(-1)^n x^n\,dx-\int\sum_{n=0}^{\infty}x^n\,dx$$

$$f(x)=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}-\sum_{n=0}^{\infty}(1)^n\frac{x^{n+1}}{n+1}$$

$$f(x)=C+\sum_{n=0}^{\infty}\left[(-1)^n-1\right]\frac{x^{n+1}}{n+1}=C+\frac{0x^1}{1}-\frac{2x^2}{2}+0\frac{x^3}{3}-2\frac{x^4}{4}+-\cdots$$

$$=C-x^2-\frac{x^4}{2}-\frac{x^6}{3}-\cdots=\boxed{C-\sum_{n=1}^{\infty}\frac{x^{2n}}{n},\ (-1,1)}$$

4. Use the power series $\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n,\ |x|<1$ to determine a power series. Identify the interval of convergence.

$$f(x)=\frac{x}{(1-x)^2}=x\left[\frac{1}{(1-x)^2}\right]=x\left[\frac{d}{dx}\left(\frac{1}{1-x}\right)\right]$$

$$f(x)=+x\frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$f(x)=+x\frac{d}{dx}\sum_{n=0}^{\infty}x^n$$

$$f(x)=+x\sum_{n=1}^{\infty}nx^{n-1}$$

$$f(x)=+\sum_{n=1}^{\infty}x^1nx^{n-1}$$

$$\boxed{f(x)=+\sum_{n=1}^{\infty}nx^n,\ (-1,1)}$$